

Advanced Quantum Mechanics Exercise Sheet 1 (due week of 19.10.2026)

Exercise 1

In a vector space $\mathcal{V}$, the following axioms have to be fulfilled. Let $|\psi\rangle$ and $|\phi\rangle$ be vectors in $\mathcal{V}$ and let α and β be scalars in $\mathbb{C}$:

- 1) $\alpha(|\psi\rangle + |\phi\rangle) = \alpha|\psi\rangle + \alpha|\phi\rangle$ and $(\alpha + \beta)|\psi\rangle = \alpha|\psi\rangle + \beta|\psi\rangle$
- 2) $\alpha(\beta|\psi\rangle) = \beta(\alpha|\psi\rangle)$
- 3) There exist a zero vector $|0\rangle \in \mathcal{V}$ such that $0 \cdot |\psi\rangle = |0\rangle, \forall |\psi\rangle \in \mathcal{V}$.
- 4) $1 \cdot |\psi\rangle = |\psi\rangle, \forall |\psi\rangle \in \mathcal{V}$.

Show, based on the above axioms, that:

$$(i) \quad |\psi\rangle + |0\rangle = |\psi\rangle$$

$$(ii) \quad \alpha(|\psi\rangle - |\phi\rangle) = \alpha|\psi\rangle - \alpha|\phi\rangle$$

Exercise 2

Let $\mathcal{V}$ be a vector space.

- (i) What are the necessary conditions on all $|\psi\rangle \in \mathcal{V}$ for $\mathcal{V}$ to be a Hilbert space?
- (ii) Let $|\psi\rangle$ be an ordered sequence of complex numbers $\psi_i \in \mathbb{C}$ where $\sum_i |\psi_i|^2 < \infty$. Is it possible to define an inner product $(\bullet, \bullet)$ according to $(|\psi\rangle, |\phi\rangle) = \sum_i \psi_i^* \phi_i$?
- (iii) Is it possible to define a Hermitian inner product $(\bullet, \bullet)$ according to $(|\psi\rangle, |\phi\rangle) = \sum_i \psi_i \phi_i$?

Exercise 3

Let $\mathcal{H}$ and $\mathcal{H}^*$ be dual Hilbert spaces. A bra vector $\langle\phi| \in \mathcal{H}^*$ corresponding to a ket vector $|\phi\rangle \in \mathcal{H}$ is defined via the inner product as $\langle\phi|\chi\rangle = (|\phi\rangle, |\chi\rangle)$. Consider $|\Psi\rangle = \alpha|\phi\rangle + \beta|\psi\rangle \in \mathcal{H}$ and $\langle\Phi| = \alpha^*\langle\phi| + \beta^*\langle\psi| \in \mathcal{H}^*$. Show that $\langle\Phi|$ corresponds to $|\Psi\rangle$, i.e., $\langle\Phi| = \langle\Psi|$.

Exercise 4

Consider the spanning set $\{|v_i\rangle\}$ of the Hilbert space $\mathcal{H}$. Prove that the sequence

$$|u_1\rangle = \frac{|v_1\rangle}{\| |v_1\rangle \|}, \quad |u_k\rangle = \frac{|v_k\rangle - \sum_{j=1}^{k-1} \langle u_j | v_k \rangle |u_j\rangle}{\| |v_k\rangle - \sum_{j=1}^{k-1} \langle u_j | v_k \rangle |u_j\rangle \|}, \quad \forall k > 1$$

forms an orthonormal basis.